

Nonperturbative QED Effective Action at Finite Temperature

Sang Pyo Kim*

*Department of Physics, Kunsan National University, Kunsan 573-701, Korea and
Asia Pacific Center for Theoretical Physics, Pohang 790-784, Korea*

Hyun Kyu Lee[†] and Yongsung Yoon[‡]

*Department of Physics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
(Dated: July 2, 2010)*

We propose a novel method for the effective action of spinor and scalar QED at finite temperature in time-dependent electric fields, where charged pairs evolve in a nonadiabatic way. The imaginary part of the effective action consists of thermal loops of the Fermi-Dirac or Bose-Einstein distribution for the initial thermal ensemble, weighted with factors of the Bogoliubov coefficients for quantum effects. And the real part of the effective action is determined by the mean number of produced pairs and vacuum polarization at zero temperature. In the weak-field limit, the mean number of produced pairs is shown twice the imaginary part. We explicitly find the finite temperature effective action in a constant electric field.

PACS numbers: 12.20.-m, 13.40.-f, 12.20.Ds, 11.15.Tk

I. INTRODUCTION

In a strong electromagnetic field the vacuum becomes polarized due to the interaction of the electromagnetic field with virtual charged pairs from the Dirac sea. The effective actions in electromagnetic fields have been continuously investigated since the early work by Sauter, Heisenberg and Euler, and Weisskopf [1] and later on the proper-time integral for the effective action by Schwinger [2]. The Heisenberg-Euler effective action exhibits both vacuum polarization and pair production, and has many physical applications (for a review and references, see Refs. [3–5]).

The effective actions at zero temperature have been a nontrivial task for general profiles of electromagnetic fields. As a strong electric field always creates pairs from the vacuum, the corresponding effective action contains not only the real part responsible for vacuum polarization but also the imaginary part for the decay of vacuum. Thus, the quantum field theory for strong electric fields should properly handle pair creation from the vacuum. The effective actions have been found for a pulsed-electric field of Sauter type in the resolvent method [6] and in the evolution operator method [7], and the effective action could be found for a spatially localized electric field [8].

However, the QED effective action at finite temperature in electric field backgrounds has been an issue of constant interest and controversy, partly because different formalisms give conflicting results [9–14] and partly because the thermal effects may be important to astrophysical objects involving strong electromagnetic fields. In fact, most methods for finite temperature field theory may not be directly applied to electric fields due to pair creation from the vacuum. The one-loop energy momentum tensor of fermions in an initial thermal ensemble was found in a constant electric field background [15]. Recently the closed-time formalism has been employed to find the QED effective action at finite temperature in 0+1 dimension [16]. The enhancement of pair production by the electric field at finite temperature is also found [17].

The purpose of this paper is two-fold: we first propose a novel method for the effective action at finite temperature for time-dependent quantum fields and then find the QED effective action in strong electric field backgrounds. At zero temperature the effective action is the scattering amplitude between the out-vacuum and the in-vacuum, which is the expectation value of the evolution operator with respect to the in-vacuum [7]. To extend the in- and out-state formalism to finite temperature, we first express the evolution operator in terms of the Bogoliubov coefficients and then find the effective action as the expectation value of the evolution operator with respect to the ‘thermal vacuum’. It turns out that the finite temperature effective action is the trace of the evolution operator weighted with the initial thermal ensemble of fermions or bosons, which is equivalent to the ‘thermal vacuum’ expectation value of the evolution operator in thermofield dynamics. The formalism may be applicable to other time-dependent quantum field, whose

*Electronic address: sangkim@kunsan.ac.kr

[†]Electronic address: hyunkyu@hanyang.ac.kr

[‡]Electronic address: cem@hanyang.ac.kr

Hamiltonian $H(t)$ evolves in a nonadiabatic way, that is, out of equilibrium, so that $e^{-\beta H(t)}$ is not the density operator satisfying the Liouville-von Neumann equation [18].

We apply the new method to the QED effective action at finite temperature in time-dependent electric fields. The QED effective action consists of the zero-temperature part, the part for thermal and vacuum fluctuations, and the finite temperature part without the electric field. The logarithm of the Bogoliubov coefficient plays a role of complex chemical potential in the complex thermal distribution for the effective action. The real and the imaginary parts of the effective action have an expansion in terms of the Fermi-Dirac or Bose-Einstein distribution and the chemical potential. Finally we find the effective action in a constant electric field and discuss it for the Sauter-type electric field.

The organization of this paper is as follows. In Sec. II, we propose a new method to find the finite temperature effective action for time-dependent systems. The effective action is given by the trace of the evolution operator and the initial density operator, which is equivalent to the expectation value of the evolution operator with respect to the thermal vacuum of thermofield dynamics. In Sec. III, we find the effective action in spinor and scalar QED in electric fields and then elaborate an expansion scheme in terms of the Fermi-Dirac and Bose-Einstein distributions. In Sec. IV, we apply the formalism to find the effective action in a constant electric field. Finally, we discuss the controversial issue of thermal effects on pair-production rate in Sec. V.

II. FINITE TEMPERATURE EFFECTIVE ACTION

We consider both spinor and scalar QED with the time-dependent gauge field $A_{\parallel}(t)$, which generates a constant or time-dependent electric field. For a pulse-like electric field acting for a finite period of time, the ingoing and the outgoing vacua are well-defined at $t_{\text{in}} = -\infty$ and $t_{\text{out}} = \infty$, for which we may choose a gauge $A_{\parallel}(t_{\text{in}}) = 0$ such that the ingoing vacuum $|0, t_{\text{in}}\rangle$ is nothing but the Minkowski vacuum $|0\rangle_{\text{M}}$. In the case of a constant electric field, we may use the asymptotic state as in Ref. [7]. The particle and antiparticle have the momentum \mathbf{k} and the spin state σ , whose annihilation operators are denoted by $a_{\mathbf{k}\sigma, \text{in}}$ and $b_{\mathbf{k}\sigma, \text{in}}$ at $t_{\text{in}} = -\infty$ and $a_{\mathbf{k}\sigma, \text{out}}$ and $b_{\mathbf{k}\sigma, \text{out}}$ at $t_{\text{out}} = \infty$, where $\sigma = \pm 1/2$ for spinor QED and $\sigma = 0$ for scalar QED. Then, the in- and out-vacua are related through the Bogoliubov transformations [19]

$$\begin{aligned} a_{\mathbf{k}\sigma, \text{out}} &= \mu_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma, \text{in}} + \nu_{\mathbf{k}\sigma}^* b_{\mathbf{k}\sigma, \text{in}}^\dagger = U_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma, \text{in}} U_{\mathbf{k}\sigma}^\dagger, \\ b_{\mathbf{k}\sigma, \text{out}} &= \mu_{\mathbf{k}\sigma} b_{\mathbf{k}\sigma, \text{in}} + \nu_{\mathbf{k}\sigma}^* a_{\mathbf{k}\sigma, \text{in}}^\dagger = U_{\mathbf{k}\sigma} b_{\mathbf{k}\sigma, \text{in}} U_{\mathbf{k}\sigma}^\dagger, \end{aligned} \quad (1)$$

where $U_{\mathbf{k}\sigma}$ is the evolution operator, whose form in terms of $\mu_{\mathbf{k}\sigma}$ and $\nu_{\mathbf{k}\sigma}$ is explicitly given in Ref. [7], and the coefficients satisfy the relation

$$|\mu_{\mathbf{k}\sigma}|^2 + (-1)^{1+2|\sigma|} |\nu_{\mathbf{k}\sigma}|^2 = 1. \quad (2)$$

In the in- and out-state formalism elaborated in Ref. [7], the in- and out-vacua are annihilated by $a_{\mathbf{k}\sigma, \text{in/out}}$ and $b_{\mathbf{k}\sigma, \text{in/out}}$. In fact, the in- and out-vacua are the tensor product of the zero-number states for all \mathbf{k} and σ . The evolution operator transforms the in-vacuum to the out-vacuum as $|0, \text{out}\rangle = U|0, \text{in}\rangle$, where U is also the tensor product of each $U_{\mathbf{k}\sigma}$, that is, $U = \prod_{\mathbf{k}\sigma} U_{\mathbf{k}\sigma}$. The zero-temperature effective action per unit volume and per unit time is obtained from the scattering amplitude [7]

$$e^{i \int d^3x dt \mathcal{L}_{\text{eff}}} = \langle 0, \text{out} | 0, \text{in} \rangle = \langle 0, \text{in} | U^\dagger | 0, \text{in} \rangle. \quad (3)$$

Now we extend the zero-temperature effective action to the finite-temperature one for the system with the initial density operator,

$$\rho_{\text{in}} = \prod_{\mathbf{k}\sigma} \left[e^{-n_{\mathbf{k}\sigma} \beta E(\mathbf{k}, \sigma)} |n_{\mathbf{k}\sigma}, \text{in}\rangle \langle n_{\mathbf{k}\sigma}, \text{in}| \right], \quad (4)$$

where $\beta = 1/k_{\text{B}}T$, k_{B} being the Boltzmann constant, and $E(\mathbf{k}, \sigma)$ is the energy for massive charged particles and antiparticles.[26]

In finite temperature field theories for static systems, one employs either the partition function $Z(\beta) = \text{Tr}(\rho)$ or the thermal expectation value $\langle O \rangle_\beta = \text{Tr}(O\rho)/\text{Tr}(\rho)$, which is equivalent to the ‘thermal vacuum’ expectation value, $\langle O \rangle_\beta = \langle 0, \beta, \text{in} | O | 0, \beta, \text{in} \rangle$ [20–22]. However, finite temperature field theories should be modified for time-dependent systems since they nonadiabatically evolve the initial states. For such time-dependent quantum fields we propose the finite-temperature effective action

$$e^{i \int d^3x dt \mathcal{L}_{\text{eff}}(T)} = \frac{\text{Tr}(U^\dagger \rho_{\text{in}})}{\text{Tr}(\rho_{\text{in}})}. \quad (5)$$

In fact, the effective action (5) is equivalent to $\langle 0, \beta, \text{in} | U^\dagger | 0, \beta, \text{in} \rangle$ for the ‘thermal vacuum’ [20, 21]

$$|0, \beta, \text{in}\rangle \equiv \frac{1}{Z_{\text{in}}^{-1/2}} \prod_{\mathbf{k}\sigma} \left[\sum_{n_{\mathbf{k}\sigma}} e^{-n_{\mathbf{k}\sigma} \beta E(\mathbf{k}, \sigma)/2} |n_{\mathbf{k}\sigma}, \text{in}\rangle \otimes |\tilde{n}_{\mathbf{k}\sigma}, \text{in}\rangle \right], \quad (6)$$

where $|\tilde{n}_{\mathbf{k}\sigma}, \text{in}\rangle$ denotes the state for a noninteracting fictitious system of the extended Hilbert space. In fact, Eq. (5) has the correct zero-temperature limit (3). The effective action (5) may be applied to other time-dependent quantum fields as well as QED, which evolve in a nonadiabatic way.

III. QED EFFECTIVE ACTION AT T

We now advance a method to compute the QED effective action (5) in electric fields. Evaluating Eq. (5), we obtain the effective action at finite temperature per unit volume and per unit time,

$$\mathcal{L}_{\text{eff}}(T, E) = (-1)^{2|\sigma|} i \sum_{\mathbf{k}\sigma} \left[-\beta z_{\mathbf{k}\sigma} + \ln(1 + (-1)^{1+2|\sigma|} e^{-\beta(\omega_{\mathbf{k}} - z_{\mathbf{k}\sigma})}) - \ln(1 + (-1)^{1+2|\sigma|} e^{-\beta\omega_{\mathbf{k}}}) \right], \quad (7)$$

where $\omega_{\mathbf{k}} = \sqrt{m^2 + \mathbf{k}_\perp^2 + (k_\parallel + qA_\parallel)^2}$ and

$$\frac{1}{\mu_{\mathbf{k}\sigma}^*} = e^{\beta z_{\mathbf{k}\sigma}}, \quad (z_{\mathbf{k}\sigma} = z_r(\mathbf{k}, \sigma) + i z_i(\mathbf{k}, \sigma)). \quad (8)$$

The summation is over all possible states such as momenta and spin states. Each term in Eq. (7) has the following interpretation: the first term is the effective action $\mathcal{L}_{\text{eff}}(T=0, E)$ at zero temperature, the second term is the combined effect of thermal and quantum fluctuations, while the last term is the subtraction of the effective action (potential energy) $\mathcal{L}_{\text{eff}}(T, E=0)$ at finite temperature without the electric field. From now on we subtract the zero-temperature part from the effective action and let

$$\Delta \mathcal{L}_{\text{eff}}(T, E) = \mathcal{L}_{\text{eff}}(T, E) - \mathcal{L}_{\text{eff}}(0, E). \quad (9)$$

Note that $z_{\mathbf{k}\sigma}(E)$, which depends on the electric field E and $z_{\mathbf{k}\sigma}(0) = 0$, plays a role of complex chemical potential, as will be explained below.

Further, we elaborate an expansion scheme for the effective action in terms of the Fermi-Dirac or Bose-Einstein distribution and $z_{\mathbf{k}\sigma}$. First, the imaginary part of the effective action (9) can be expanded as

$$\text{Im}(\Delta \mathcal{L}_{\text{eff}}) = (-1)^{1+2|\sigma|} \frac{1}{2} \sum_{\mathbf{k}\sigma} \sum_{j=1}^{\infty} \frac{[(-1)^{2|\sigma|} n_{F/B}(\mathbf{k})]^j}{j} [(e^{\beta z_{\mathbf{k}\sigma}} - 1)^j + (e^{\beta z_{\mathbf{k}\sigma}^*} - 1)^j], \quad (10)$$

where $n_{F/B}(\mathbf{k})$ denotes either the Fermi-Dirac distribution $n_F(\mathbf{k}) = 1/(e^{\beta\omega_{\mathbf{k}}} + 1)$ for spinor QED or $n_B(\mathbf{k}) = 1/(e^{\beta\omega_{\mathbf{k}}} - 1)$ for scalar QED. Second, the real part of the effective action (9) is given by

$$\text{Re}(\Delta \mathcal{L}_{\text{eff}}) = (-1)^{2|\sigma|} \sum_{\mathbf{k}\sigma} \sum_{j=1}^{\infty} \frac{[(-1)^{2|\sigma|} e^{-\beta(\omega_{\mathbf{k}} - z_r(\mathbf{k}, \sigma))}]^j}{j} \sin(j\beta z_i(\mathbf{k}, \sigma)). \quad (11)$$

The thermal factors in Eqs. (10) and (11) correspond to thermal loops in the diagrammatic representation, which are weighted with factors from quantum fluctuations

Now, we give physical interpretations for the effective action. In the weak-field limit ($qE \ll m^2$) where $\beta z_i(\mathbf{k}) \ll 1$, the real part (11) approximately is

$$\text{Re}(\Delta \mathcal{L}_{\text{eff}}) \approx \sum_{\mathbf{k}\sigma} \frac{\beta z_i(\mathbf{k}, \sigma)}{e^{\beta(\omega_{\mathbf{k}} - z_r(\mathbf{k}, \sigma))} + (-1)^{1+2|\sigma|}}, \quad (12)$$

while the imaginary part (10) approximately leads to

$$2\text{Im}(\Delta \mathcal{L}_{\text{eff}}) \approx (-1)^{2|\sigma|} \sum_{\mathbf{k}\sigma} |\nu_{\mathbf{k}\sigma}|^2 n_{F/B}(\mathbf{k}). \quad (13)$$

Thus, the imaginary part may be regarded as the pair-production rate due to thermal and quantum effects. The thermal effects suppress the fermion pair production due to the Pauli blocking but enhance the boson pair production due to the Bose-Einstein condensation, as expected. In Ref. [19], the mean number of produced pairs with a given momentum \mathbf{k} at T is given by $\bar{N}^{\text{sp}}(T) = \sum_{\mathbf{k}\sigma} |\nu_{\mathbf{k}\sigma}|^2 \tanh(\beta\omega_{\mathbf{k}}/2)$ for spinor QED and $\bar{N}^{\text{sc}}(T) = \sum_{\mathbf{k}\sigma} |\nu_{\mathbf{k}\sigma}|^2 \coth(\beta\omega_{\mathbf{k}}/2)$ for scalar QED. So the mean number, $\Delta\bar{N} = (\bar{N}(T) - \bar{N}(0))/2$, of one species of particle or antiparticle due to thermal effects approximately satisfies the relation between the mean number and the imaginary part:

$$\Delta\bar{N} = \sum_{\mathbf{k}\sigma} |\nu_{\mathbf{k}\sigma}|^2 n_{F/B}(\mathbf{k}) \approx 2\text{Im}(\Delta\mathcal{L}_{\text{eff}}). \quad (14)$$

The relation between the mean number of produced pairs and twice of the imaginary part also holds at $T = 0$ in the weak-field limit [7].

A few comments are in order. The series of the real part (11) may be summed as [23]

$$\text{Re}(\Delta\mathcal{L}_{\text{eff}}) = \sum_{\mathbf{k}\sigma} \arctan \left[\frac{\sin(\beta z_i(\mathbf{k}))}{e^{\beta(\omega_{\mathbf{k}} - z_r(\mathbf{k}))} + (-1)^{1+2|\sigma|} \cos(\beta z_i(\mathbf{k}))} \right]. \quad (15)$$

Using $\mathcal{L}_{\text{eff}}(0, E) = (-1)^{1+2|\sigma|} i \sum_{\mathbf{k}\sigma} \beta z_{\mathbf{k}\sigma}$ from Eq. (7) and the Bogoliubov relation (2), we have the real and imaginary parts

$$\begin{aligned} \beta z_r(\mathbf{k}, \sigma) &= (-1)^{1+2|\sigma|} \text{Im}(\mathcal{L}_{\text{eff}}(0, E)) = -\frac{1+2|\sigma|}{2} \ln(1 + (-1)^{2|\sigma|} |\nu_{\mathbf{k}\sigma}|^2), \\ \beta z_i(\mathbf{k}, \sigma) &= (-1)^{2|\sigma|} \text{Re}(\mathcal{L}_{\text{eff}}(0, E)). \end{aligned} \quad (16)$$

Then the effective action (15) at finite temperature can be written in terms of the mean number, the vacuum polarization, and the thermal distribution as

$$\text{Re}(\Delta\mathcal{L}_{\text{eff}}) = (-1)^{2|\sigma|} \sum_{\mathbf{k}\sigma} \arctan \left[\frac{\sin(\text{Re}(\mathcal{L}_{\text{eff}}(0, E)))}{e^{\beta\omega_{\mathbf{k}}} (1 + (-1)^{2|\sigma|} |\nu_{\mathbf{k}\sigma}|^2)^{\frac{1+2|\sigma|}{2}} + (-1)^{1+2|\sigma|} \cos(\text{Re}(\mathcal{L}_{\text{eff}}(0, E)))} \right]. \quad (17)$$

Another interesting observation is that the mid-term in Eq. (7),

$$W_{\text{eff}}(T, E) = (-1)^{1+2|\sigma|} i \sum_{\mathbf{k}\sigma} \ln(1 + (-1)^{1+2|\sigma|} e^{-\beta(\omega_{\mathbf{k}} - z_{\mathbf{k}\sigma})}), \quad (18)$$

is reminiscent of the potential energy [24] and carries both thermal and quantum effects. Equation (18) suggests $z_{\mathbf{k}\sigma}$ as the chemical potential and the variation with respect to $z_{\mathbf{k}\sigma}$ yields the Fermi-Dirac or Bose-Einstein distribution.

IV. APPLICATIONS

In this section we find the QED effective action in a constant electric field and discuss a Sauter-type electric field, $E(t) = E_0 \text{sech}^2(t/\tau)$ in Ref. [7]. In the time-dependent gauge, $A_{\parallel}(t) = -Et$ for the constant electric field and $A_{\parallel}(t) = -E_0\tau(1 + \tanh(t/\tau))$ for the Sauter-type electric field, the energy of charged particles explicitly depend on time. We will take the weak-field limit ($qE \ll m^2$), where the real part (12) and the imaginary part (13) of the approximate effective action can be worked out for the constant electric field and in principle for the Sauter-type electric field.

In the constant electric field, the state along the direction of the electric field is asymptotically determined, whose momentum integral gives a factor $qE/(2\pi)$ [7]. Using the mean number of produced pairs,

$$|\nu_{\mathbf{k}\sigma}|^2 = e^{-\pi \frac{m^2 + \mathbf{k}^2}{qE}}, \quad (19)$$

which is independent of the spin states, the imaginary part (13) is given by

$$\begin{aligned} \text{Im}(\Delta\mathcal{L}_{\text{eff}}(T, E)) &\approx \frac{1+2|\sigma|}{2} \left(\frac{qE}{2\pi} \right)^2 e^{-\frac{\pi m^2}{qE}} \sum_{n=0}^{\infty} (-1)^{2|\sigma|(n+1)} \frac{m^2 e^{-\beta m(n+1)}}{2\pi m^2 + \beta m q E (n+1)} \\ &\times \left[1 + \frac{\beta m (qE)^2}{(2\pi m^2 + \beta m q E (n+1))^2} - \frac{3\beta m (qE)^3}{(2\pi m^2 + \beta m q E (n+1))^3} + \cdots \right]. \end{aligned} \quad (20)$$

The factor in front of the summation is the leading term of the imaginary part at zero temperature. Further, in the low-temperature limit ($\beta m \gg 1$), the leading term of Eq. (20) is

$$\text{Im}(\Delta\mathcal{L}_{\text{eff}}(T, E)) \approx \frac{1 + 2|\sigma|}{2} \left(\frac{qE}{2\pi}\right)^2 e^{-\frac{\pi m^2}{qE}} \left[(-1)^{2|\sigma|} \frac{m^2}{qE} \frac{e^{-\beta m}}{\beta m + \frac{2\pi m^2}{qE}}\right]. \quad (21)$$

In the special case of thermal effect dominance, neglecting all terms of $m/\beta qE$, the first series in Eq. (20) approximately leads to

$$\text{Im}(\Delta\mathcal{L}_{\text{eff}}(T, E)) \approx \frac{1 + 2|\sigma|}{2} \left(\frac{qE}{2\pi}\right)^2 e^{-\frac{\pi m^2}{qE}} \left[-\frac{m^2}{\beta m} \ln(1 + (-1)^{1+2|\sigma|} e^{-\beta m})\right]. \quad (22)$$

Similarly, using $\text{Re}(\mathcal{L}_{\text{eff}}(0, E))$ in Ref. [7], the real part (12), for instance, of spinor QED is given by

$$\begin{aligned} \text{Re}(\Delta\mathcal{L}_{\text{eff}}^{\text{sp}}(T, E)) \approx & -\frac{qE}{2\pi} \frac{m^2}{2\pi} \sum_{n=0}^{\infty} (-1)^n \sum_{l=2}^{\infty} \frac{2^{4l-2} |B_{2l}|}{(2l)!} \left(\frac{qE}{2\pi}\right)^{2l-1} \frac{1}{m^{4l-2}} \\ & \times \left[e^{-\beta m(n+1)} \left(\Psi(1, 3-2l, \alpha) + \frac{\beta m(n+1)}{4} \Psi(3, 5-2l, \alpha) - \frac{3\beta m(n+1)}{8} \Psi(4, 6-2l, \alpha) + \dots \right) \right. \\ & \left. - e^{-\beta mn} \left(\Psi(1, 3-2l, \gamma) + \frac{\beta mn}{4} \Psi(3, 5-2l, \gamma) - \frac{3\beta mn}{8} \Psi(4, 6-2l, \gamma) + \dots \right) + \dots \right], \end{aligned} \quad (23)$$

where B_{2l} is the Bernoulli number, Ψ denotes the second confluent hypergeometric function [25], and

$$\alpha = \frac{\beta m(n+1)}{2}, \quad \gamma = \frac{\beta m(n+1)}{2} + \frac{\pi m^2}{qE}. \quad (24)$$

In the low-temperature limit ($\beta m \gg 1$), the series of $l = 2$ in Eq. (23) leads to

$$\text{Re}(\Delta\mathcal{L}_{\text{eff}}^{\text{sp}}(T, E)) \approx -\frac{(2\pi)^2}{45m^4} \left(\frac{qE}{2\pi}\right)^4 \left[(-1) \frac{3}{\beta m} \ln(1 + e^{-\beta m})\right]. \quad (25)$$

Here the factor in front of the square bracket is the real part at zero temperature. The real part of effective action in scalar QED may be found in a similar way.

Finally, we discuss the Sauter-type electric field. The charged particle has the free energy $\omega_{\mathbf{k}, \text{in}} = \sqrt{m^2 + \mathbf{k}^2}$ before the onset of the electric field while it has $\omega_{\mathbf{k}, \text{out}} = \sqrt{m^2 + \mathbf{k}_{\perp}^2 + (k_z - 2qE_0\tau)^2}$ after the completion of the interaction. At zero temperature, the mean number of produced pairs, Eqs. (68) and (83), and the vacuum polarization, Eqs. (66) and (80) of Ref. [7], which depend on $\omega_{\mathbf{k}, \text{in}}$, $\omega_{\mathbf{k}, \text{out}}$, and $\lambda = \sqrt{(qE_0\tau^2)^2 - (2|\sigma| - 1)^2/4}$, lead to the effective action, Eqs. (12) and (13). To find analytical expressions for the effective action would be more complicated than the constant electric field, which will be addressed elsewhere.

V. CONCLUSION

In this paper we have advanced a new method for the finite-temperature effective action for time-dependent quantum fields and have studied the one-loop effective action of spinor and scalar QED at finite temperature in a constant or time-dependent electric fields. External electric fields make the vacuum unstable against pair production, which is a consequence of the out-vacuum differing from the in-vacuum. The instability enforces a careful application of the finite-temperature field theory to time-dependent quantum fields. The finite-temperature effective action (5) is given by the trace of the initial thermal ensemble evolved by the time-evolution operator, which is equivalent to the thermal vacuum expectation value of the evolution operator in thermofield dynamics.

The imaginary part (10) of the effective action exhibits factorization into thermal factors and quantum factors, which correspond to thermal loops in the diagrammatic representation with vertices of the external electric field. In the weak-field limit ($qE \ll m^2$), twice of the imaginary part is the mean number of produced pairs, as shown in Eq. (13). However, the thermal and quantum effects are intertwined in the real part of the effective action, Eqs. (11), (15), and (17). In fact, the finite-temperature effective action (17) is determined by the vacuum polarization at zero temperature, the mean number of produced pairs, and thermal distribution. In the weak-field and lower-temperature limits, the leading factors of the real and imaginary parts, (22) and (25), of the effective action in a constant electric field are proportional to those at zero temperature and the potential energy for the rest mass in spinor and scalar QED.

Our results show many interesting aspects. First, the imaginary part (10) of the thermal contribution does not vanish for any non-zero electric field, which implies thermal effects on pair production and thus may resolve the controversial issue of thermal effects on pair production: thermal effects are shown to exist in Refs. [10, 13], while in Refs. [11] no thermal effects are found. Further, in the weak-field limit for small pair production, twice of the imaginary parts (14) are the pair-production rate at T , which was shown in Ref. [19]. Though our in- and out-state formalism differs from the imaginary-time formalism, the imaginary part (13) in the weak-field limit is the pair-production rate times the Fermi-Dirac or Bose-Einstein distribution, which may correspond to two-loop dominance in Ref. [14]. Second, the Bogoliubov coefficient (8), which is responsible for vacuum polarization at $T = 0$, plays a role of chemical potential in the effective action (7) and in the potential energy (18) at T . In fact, the variation of the effective action with respect to the chemical potential yields the Fermi-Dirac or the Bose-Einstein distribution.

Acknowledgments

The authors would like to thank Holger Gies for useful information and comments, Ismail Zahed for useful discussions, and Sergey Gavrilov for useful comments. S. P. K. would like to thank W-Y. Pauchy Hwang for the warm hospitality at National Taiwan University, where part of this paper was written. The work of S. P. K. was supported by the Korea Research Foundation (KRF) Grant funded by the Korea Ministry of Education, Science and Technology (2009-0075-773) and the work of H. K. L. was supported by the World Class University Program (R33-2008-000-10087-0) of the Korea Ministry of Education, Science and Technology.

-
- [1] F. Sauter, Z. Phys. **69**, 742 (1931); W. Heisenberg and H. Euler, Z. Physik **98**, 714 (1936); V. Weisskopf, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **XIV**, No. 6 (1936).
 - [2] J. Schwinger, Phys. Rev. **82**, 664 (1951).
 - [3] W. Dittrich and H. Gies, Springer Tracts Mod. Phys. **166**, 1 (2000).
 - [4] G. V. Dunne, “Heisenberg-Euler Effective Lagrangians: Basics and Extensions,” *From Fields to Strings: Circumnavigating Theoretical Physics*, edited by M. Shifman, A. Vainshtein, and J. Wheeler, (World Scientific, Singapore, 2005), Vol. I, pp. 445-522, hep-th/0406216.
 - [5] R. Ruffini, G. Vereshchagin, and S.-S. Xue, Phys. Rep. **487**, 1 (2010).
 - [6] G. V. Dunne and T. Hall, Phys. Rev. D **58**, 105022 (1998).
 - [7] S. P. Kim, H. K. Lee, and Y. Yoon, Phys. Rev. D **78**, 105013 (2008).
 - [8] S. P. Kim, H. K. Lee, and Y. Yoon, “Effective Action of QED in Electric Field Backgrounds II: Spatially Localized Fields,” [arXiv:hep-th/0910.3363].
 - [9] P. Elmfors, D. Persson, and B.-S. Skagerstam, Phys. Rev. Lett. **71**, 480 (1993); P. Elmfors, P. Liljeborg, D. Persson, and B.-S. Skagerstam, Phys. Rev. D **51**, 5885 (1995).
 - [10] M. Loewe and J. C. Rojas, Phys. Rev. D **46**, 2689 (1992).
 - [11] P. Elmfors and B.-S. Skagerstam, Phys. Lett. B **348**, 141 (1995); Erratum, Phys. Lett. B **376**, 330 (1996).
 - [12] A. K. Ganguly, P. K. Kaw, and J. C. Parikh, Phys. Rev. C **51**, 2091 (1995).
 - [13] J. Hallin and P. Liljeborg, Phys. Rev. D **52**, 1150 (1995).
 - [14] H. Gies, Phys. Rev. D **60**, 105002 (1999); Phys. Rev. D **61**, 085021 (2000).
 - [15] S. P. Gavrilov and D. M. Gitman, Phys. Phys. D **78**, 045017 (2008).
 - [16] A. Das and J. Frenkel, Phys. Lett. B **680**, 195 (2009); Phys. Rev. D **80**, 125039 (2009).
 - [17] A. K. Monin and A. V. Zayakin, JETP Lett. **87**, 709 (2008); A. Monin and M. B. Voloshin, Phys. Rev. D **81**, 025001 (2010).
 - [18] S. P. Kim and C. H. Lee, Phys. Rev. D **62**, 125020 (2000).
 - [19] S. P. Kim and H. K. Lee, Phys. Rev. D **76**, 125002 (2007); S. P. Kim, H. K. Lee, and Y. Yoon, Phys. Rev. D **79**, 045024 (2009).
 - [20] Y. Takahashi and H. Umezawa, Collect. Phenom. **2**, 55 (1975) [reprinted in Int. J. Mod. Phys. B **10** (1996), 1755].
 - [21] H. Umezawa, *Advanced Field Theory: Micro, Macro and Thermal Physics* (AIP, New York, 1993).
 - [22] A. Das, *Finite Temperature Field Theory*, (World Scientific, Singapore, 1997).
 - [23] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach Science Publishers, The Netherlands, 1998) Vol. 1, formula 5.4.9-12.
 - [24] J. I. Kapusta, *Finite-temperature field theory* (Cambridge University Press, New York, 1989).
 - [25] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach Science Publishers, The Netherlands, 1998) Vol. 1, formula 2.3.6-9, which reads $\int_0^\infty \frac{x^{a-1}e^{-px}}{(x+z)^b} dx = \Gamma(a)z^{a-b}\Psi(a, a+1-b, pz)$.
 - [26] The unit system of $c = \hbar = k_B = 1$ is used, where qE/m^2 and βm are dimensionless in QED.